Md Tariqul Islam

Ans- 1

$f(x) = x^3 + 2x^2 + 10x - 20 = 0$

$f'(x) = 3x^2 + 4x + 10$

$x^0 = \sqrt[3]{20}$

$$x^1 = x^0 - \frac{f(x)}{f'(x)}$$

$$x^1 = \sqrt[3]{20} - \frac{(\sqrt[3]{20})^3 + 2(\sqrt[3]{20})^2 + 10(\sqrt[3]{20}) - 20}{3(\sqrt[3]{20})^2 + 4(\sqrt[3]{20}) + 10}$$

$x^1 = 1.739592$

$$x^2 = 1.739592 - \frac{(1.739592)^3 + 2(1.739592)^2 + 10(1.739592) - 20}{3(1.739592)^2 + 4(1.739592) + 10}$$

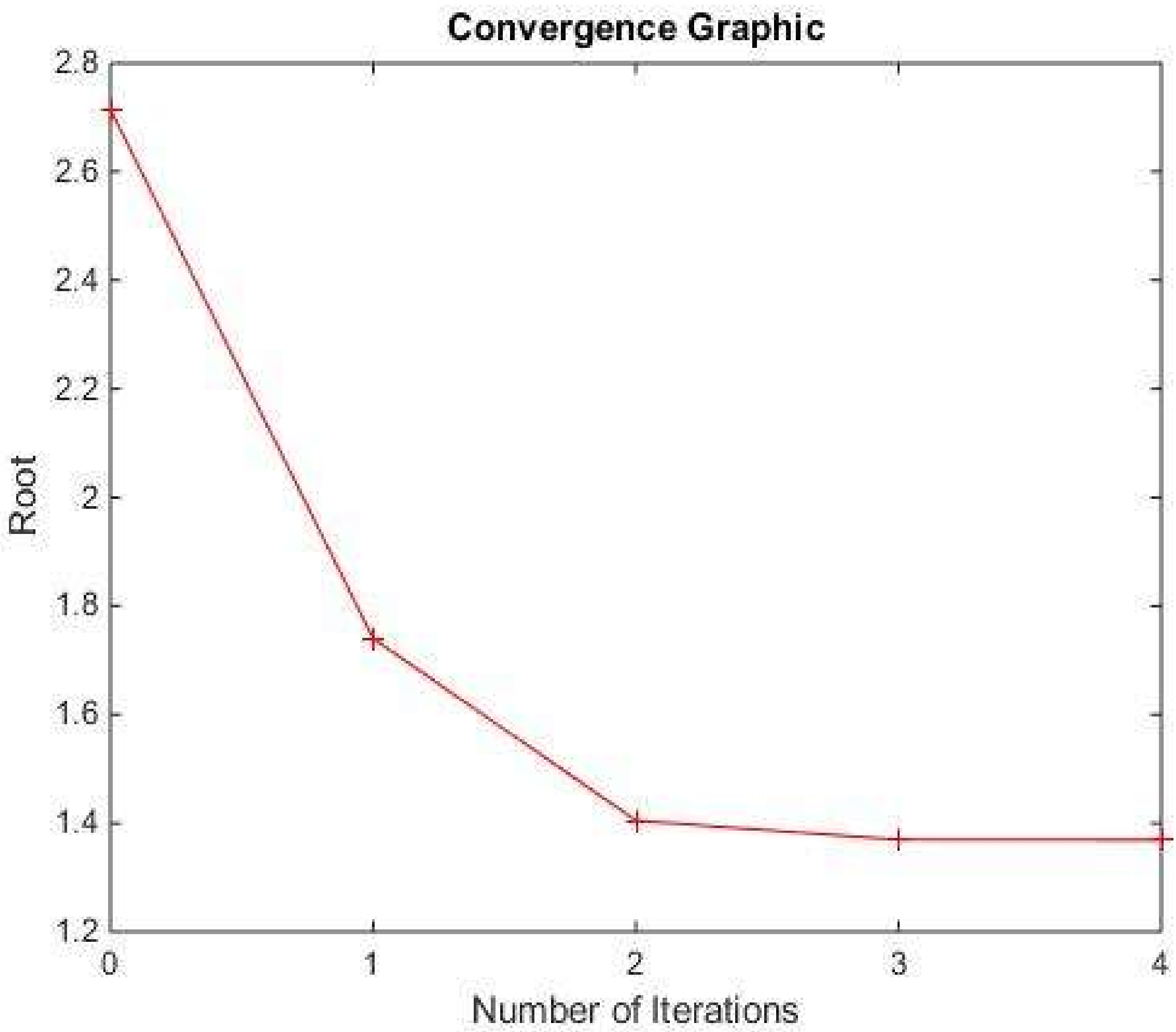
$x^2 = 1.404967$

$$x^3 = 1.404967 - \frac{(1.404967)^3 + 2(1.404967)^2 + 10(1.404967) - 20}{3(1.404967)^2 + 4(1.404967) + 10}$$

$x^3 = 1.369185$

$$x^4 = 1.369185 - \frac{(1.369185)^3 + 2(1.369185)^2 + 10(1.369185) - 20}{3(1.369185)^2 + 4(1.369185) + 10}$$

$x^4 = 1.3688$ //


Convergence Graphic
Root
Number of Iterations
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
0
1
2
3
4

Ans - 5

a) For third-order numerical quadrature, the number of integration points are:

$$2n + 1 = 3$$
$$2n = 2$$
$$n = 1$$

we need 2 points.

The integration points are:-

$$x_0 = \frac{b-a}{2} z + \frac{b+a}{2}$$

$$= \frac{1-0}{2}(-0.577) + \frac{1+0}{2}$$

$$= 0.2115$$

$$x_1 = \frac{b-a}{2} z + \frac{b+a}{2}$$

$$= \frac{1-0}{2}(0.577) + \frac{1+0}{2}$$

$$= 0.7885$$

$\therefore x_0 = 0.2115 \qquad x_1 = 0.7885$

The weights are:

$$\omega_0 = 1\left(\frac{b-a}{2}\right) = 1\left(\frac{1-0}{2}\right) = \frac{1}{2}$$

$$\omega_1 = 1\left(\frac{b-a}{2}\right) = 1\left(\frac{1-0}{2}\right) = \frac{1}{2}$$

$\therefore \omega_0 = \frac{1}{2}, \qquad \omega_1 = \frac{1}{2}$

b) It is possible to obtain a third-order quadrature by a polynomial of order 3. The corresponding weights can be calculated as follows:

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

$$\Rightarrow \int_0^1 (Ax^3 + Bx^2 + Cx + D)\,dx$$

$$\Rightarrow \left.\frac{Ax^4}{4} + \frac{Bx^3}{3} + \frac{Cx^2}{2} + Dx\right|_0^1 = \omega_0 f(\tfrac{1}{4}) + \omega_1 f(\tfrac{1}{2}) + \omega_2 f(\tfrac{3}{4}) + \omega_3 f(1)$$

$$\Rightarrow \frac{A}{4} + \frac{B}{3} + \frac{C}{2} + D = \omega_0\left(\frac{A}{64} + \frac{B}{8} + \frac{C}{4} + D\right) + \omega_1\left(\frac{A}{8} + \frac{B}{4} + \frac{C}{2} + D\right)$$
$$+ \omega_2\left(\frac{27}{64}A + \frac{9}{16}B + \frac{3}{4}C + D\right) + \omega_3(A + B + C + D)$$

$$\Rightarrow A\left(\frac{\omega_0}{64} + \frac{\omega_1}{8} + \frac{27}{64}\omega_2 + \omega_3 - \frac{1}{4}\right) + B\left(\frac{\omega_0}{8} + \frac{\omega_1}{4} + \frac{9}{16}\omega_2 + \omega_3 - \frac{1}{3}\right)$$
$$+ C\left(\frac{\omega_0}{4} + \frac{\omega_1}{2} + \frac{3}{4}\omega_2 + \omega_3 - \frac{1}{2}\right) + D(\omega_0 + \omega_1 + \omega_2 + \omega_3 - 1) = 0$$

$$\therefore \frac{\omega_0}{64} + \frac{\omega_1}{8} + \frac{27}{64}\omega_2 + \omega_3 - \frac{1}{4} = 0 \quad \cdots (1)$$

$$\frac{\omega_0}{8} + \frac{\omega_1}{4} + \frac{9}{16}\omega_2 + \omega_3 - \frac{1}{3} = 0 \quad \cdots (2)$$

$$\frac{\omega_0}{4} + \frac{\omega_1}{2} + \frac{3}{4}\omega_2 + \omega_3 - \frac{1}{2} = 0 \quad \cdots (3)$$

$$\omega_0 + \omega_1 + \omega_2 + \omega_3 - 1 = 0 \quad \cdots (4)$$

Solving 4 equation for four variables, we get the weights:

$\omega_0 = 0.6667$ $\quad\omega_1 = -0.3333$ $\quad\omega_2 = 0.6667$ $\quad\omega_3 = 0$

Ans - 6

a) For n+1 points, the order of polynomial that can be integrated exactly is:

$$2n + 1$$

b) When n = 2, polynomials of degree up to 5 are integrated exactly.

$\therefore \int_0^1 x^3 dx$ & $\int_0^1 x^4 dx$ can only be integraded exactly.

Ans - 7

1) $\int_0^1 12x\,dx$

0   1

x_0 x_1 x_2

Using Trapezoidal rule :-

$$I = \frac{h}{2}\left(f(x_0)+f(x_1)\right)+\frac{h}{2}\left(f(x_1)+f(x_2)\right)$$

$$= \frac{h}{2}\left(f(x_0)+2f(x_1)+f(x_2)\right)$$

$$= \frac{0.5}{2}\left(f(0)+2f(0.5)+f(1)\right)$$

$$= \frac{0.5}{2}\left(0+2\times 6+12\right)$$

$$= 6$$

$h = \frac{1-0}{2}$

$= 0.5$

$x_0 = 0$

$x_1 = 0.5$

$x_2 = 1$

$$\text{error} = E = -\frac{h^3}{12}f''(\xi) = -\frac{(0.5)^3}{12}\times 0(\xi) = 0$$

$\int_0^1 (5x^3+2x)\,dx$

$$I = \frac{h}{2}\left(f(x_0)+f(x_1)\right)+\frac{h}{2}\left(f(x_1)+f(x_2)\right)$$

$$= \frac{h}{2}\left(f(x_0)+2f(x_1)+f(x_2)\right)$$

$$= \frac{0.5}{2}\left(f(0)+2f(0.5)+f(1)\right)$$

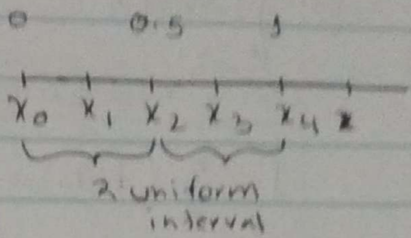
$$= \frac{0.5}{2}\left(0+2(1.625)+7\right)$$

$$= 2.5625$$

$$\text{error} = E = -\frac{h^3}{12}f''(\xi) = -\frac{(0.5)^3}{12}30x(\xi),\ \xi\in(0,1)$$

ii) Using Simpson's rule

$$\int_0^1 12x\,dx$$



$$I = \frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right)+\frac{h}{3}\left(f(x_2)+4f(x_3)+f(x_4)\right)$$

$$= \frac{0.25}{3}\left(f(0)+4f(0.25)+f(0.5)\right)+\frac{0.25}{3}\left(f(0.5)+4f(0.75)+f(1)\right)$$

$$= \frac{0.25}{3}(0+12+6)+\frac{0.25}{3}(6+36+12)$$

$$= 1.5 + 4.5$$

$$= \underline{\underline{6}}$$

$h = \frac{0.5-0}{2}$

$= 0.25$

$x_0 = 0$

$x_1 = 0.25$

$x_2 = 0.5$

$x_3 = 0.75$

$x_4 = 1$

$$\text{error} = E = -\frac{h^5}{90} f''''(\xi) = 0$$

$$\int_0^1 (5x^3+2x)\,dx$$

$$I = \frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right)+\frac{h}{3}\left(f(x_2)+4f(x_3)+f(x_4)\right)$$

$$= \frac{0.25}{3}\left(f(0)+4f(0.25)+f(0.5)\right)+\frac{0.25}{3}\left(f(0.5)+4f(0.75)+f(1)\right)$$

$$= \frac{0.25}{3}(0+2.3125+1.625)+\frac{0.25}{3}(1.625+14.4375+7)$$

$$= \underline{\underline{2.25}}$$

$$\text{error} = E = -\frac{h^5}{90} f''''(\xi) = 0$$

The methods behaved exactly as expected. Simpson's gave exact results without any error till third degree polynomial. Trapezoid didn't give any error in case of linear function, but some error in case of higher degree polynomial.

Ans - 10

As the integral eq in seperable, we will apply simpson's individually in each direction.

$h = \frac{0-1}{2}$

$= 0.5$

$x_0 = 0$

$x_1 = 0.5$

$x_2 = 1$

$$\int_0^1 (9x^3 + 8x^2)\,dx$$

$$I = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$

$$= \frac{0.5}{3}\left(0 + 4(3.125) + 17\right)$$

$$= 4.9166$$

$$\int_0^1 y^3 + y\,dy$$

$$I = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$

$$= \frac{0.5}{3}\left(0 + 4(0.625) + 2\right)$$

$$= 0.75$$

$$\therefore I = 0.75 \times 4.9166$$

$$= 3.6875$$

$$E = \text{error} = -\frac{h^5}{90} f''''(\xi) = 0$$

The approximation behaved as expected giving exact value without any error for third degree polynomial.