Numerical Methods for Partial Differential Equations

Basics

Computational Mechanics 2016-2017

Mohammad Mohsen Zadehkamand

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1*. In 1225, Leonardo of Pisa (also known as Fibonacci) was requested to solve a collection of mathematical problems in order to justify his fame and prestige in the court of Federico II. One of the proposed problems can be formulated as the solution of a third degree polynomial equation (1)

$$\boldsymbol{f(x) := x^3 + 2x^2 + 10x - 20 = 0}$$

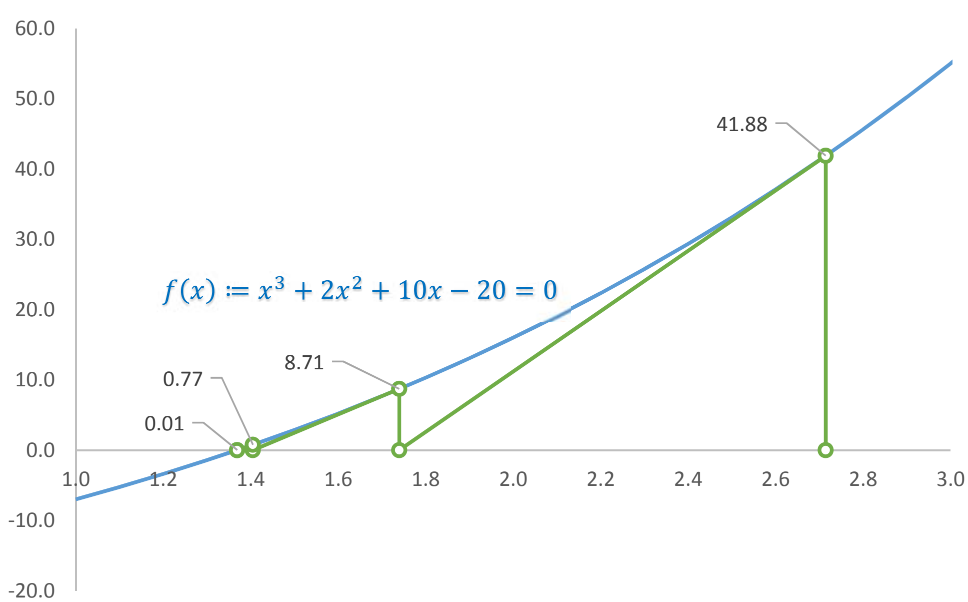
Note that the solution of cubic equations was a extremely difficult problem in the 13th century. Here iterative methods are considered for the solution of equation (1).

Compute the unique real root of (1) with 4 iterations of Newton's method with the initial approximation $\boldsymbol{x^0} = \sqrt[3]{\boldsymbol{20}}$ (which is obtained neglecting the monomials with x and x2 in front of the monomial with $\boldsymbol{x^3}$). Plot the convergence graphic. Does Newton's method behave as expected?

$$\Delta x^{k+1} = -\frac{f(x^k)}{f'(x^k)}$$

$$x^{k+1} = x^k + \Delta x^{k+1}$$

		f(x)	f'(x)
x0	2.71	41.88	42.96
x1	1.74	8.71	26.04
x2	1.40	0.77	21.54
x3	1.37	0.01	21.10



5*. We are interested in the definition of third-order numerical quadratures in interval (0; 1)

a) Determine the minimum number of integration points, and specify the integration points and weights.

b) Is it possible to obtain a third-order quadrature with the following four integration points: x0=1/4, x1=1/2, x2=3/4 and x3=1? If it is possible, compute the corresponding weights; otherwise, justify why not.

a) 2n+1=3 →n=1 so 2 integration points would be needed.

$$\int_{-1}^{+1} f(z)dz \approx \sum_{i=0}^{i=n} \omega_i f(z_i) \qquad \int_a^b F(x)dx \approx \frac{b-a}{2}\sum_{i=0}^{i=n} \omega_i F\left(\frac{(b-a)z_i+(b+a)}{2}\right)$$

n	i	z_i	ω_i
0	0	0.00000000000000E+00	0.20000000000000E+01
1	0	-0.57735026918963E+00	0.10000000000000E+01
	1	0.57735026918963E+00	0.10000000000000E+01

But here we need to change the domain boundaries to 0 and 1:

$$\int_0^1 F(x)\ dx \approx \frac{1-0}{2}\sum_{i=0}^{i=1} w_i F\left(\frac{(b-a)z_i+(b+a)}{2}\right) = \frac{1}{2}\sum_{i=0}^{i=1} w_i F(\frac{z_i+1}{2})$$

$$w_0 = \frac{1}{2}\ ,\ z_0 = 2*(-0.57735026918963) - 1 = -2.154700538E+00$$

$$w_1 = \frac{1}{2}\ ,\ z_1 = 2*(0.57735026918963) - 1 = 0.1547005384E+00$$

b)

$$\int_0^1 (Ax^3+Bx^2+Cx+D)\ dx = w_0 f\left(\frac{1}{4}\right) + w_1 f\left(\frac{1}{2}\right) + w_2 f\left(\frac{3}{4}\right) + w_3 f(1)$$

$$Right\ Side = \frac{A}{4}x^4 + \frac{B}{3}x^3 + \frac{C}{2}x^2 + Dx = \frac{A}{4} + \frac{B}{3} + \frac{C}{2} + D = Left\ side$$

After rearranging we would have:

$$A * (0.25) + B * (0.333) + C * (0.5) + D * (1) = A(P1) + B(P2) + C(P3) + D(P4)$$

Then we would have 4 equations and 4 unknowns for w0, w1, w2, w3.

$$P1: \ (\frac{1}{4})^3 w_0 + (\frac{1}{2})^3 w_1 + (\frac{3}{4})^3 w_2 + w_3 = {}^1\!/_4$$

$$P2: \ (\frac{1}{4})^2 w_0 + (\frac{1}{2})^2 w_1 + (\frac{3}{4})^2 w_2 + w_3 = {}^1\!/_3$$

$$P3: \ (\frac{1}{4}) \ \ w_0 + (\frac{1}{2}) \ \ w_1 + (\frac{3}{4}) \ \ w_2 + w_3 = {}^1\!/_2$$

$$P4: \ w_0 + w_1 + w_2 + w_3 = 1$$

By solving we would have:

$$w_0 = 2.6666$$

$$w_1 = -3.333$$

$$w_2 = 1.6666$$

$$w_3 = 0.0000$$

$$\int_0^1 F(x) \ dx \approx \frac{1-0}{2} \sum_{i=0}^{i=3} w_i F\left(\frac{(b-a)z_i + (b+a)}{2}\right)$$

$$= \frac{1}{2} \sum_{i=0}^{i=3} w_i F(\frac{z_i + 1}{2})$$

6*. a) If n + 1 points Gaussian quadrature is used for numerical integration state the order of the polynomial that is integrated exactly.

b) If n = 2 is selected for Gaussian quadrature, which (if any) of the following integrals will be integrated exactly?

$i) \int_0^1 \sin x \, dx$

$ii) \int_0^1 x^3 \, dx$

$iii) \int_0^1 x^4 dx$

$iv) \int_0^1 x^{5.5} dx$

a) Gauss Shows the exactness of polynomials of degree not exceeding 2n-1 for n nodes. Then for n+1 nodes we would have 2(n+1)-1=2n+1 as the order of exactness for polynomials

b) For m=2 (m is the subdivision parts) we have n+1=3 nodes and the order of exactness would be 2n-1=5.

(i) is not a polynomial and is not appropriate for adjusting gauss.

(ii) order of polynomial is less than 5 and could be integrated exactly.

(iii) order of polynomial is less than 5 and could be integrated exactly.

(iv) order of polynomial is more than 5 and could not be integrated exactly.

7*. Compute $\int_0^1 12x\, dx$ and $\int_0^1 (5x^3 + 2x)\, dx$ by hand calculation using

i) Trapezoidal rule over 2 uniform intervals

ii) Simpson's rule over 2 uniform intervals

Compute the error of both approximations. Are the methods behaving as expected?

i)

$$I_{Trapezoidal} = \int_0^1 f(x)dx \approx h\left[\frac{1}{2}f(a) + f(x1) + \frac{1}{2}f(b)\right], h = \frac{b-a}{n}$$

$$I_i \approx 0.5 * \left[\frac{1}{2}f(0) + f(0.5) + \frac{1}{2}f(1)\right] = 6.0000\,, h = \frac{1-0}{2} = 0.5, n = 2$$

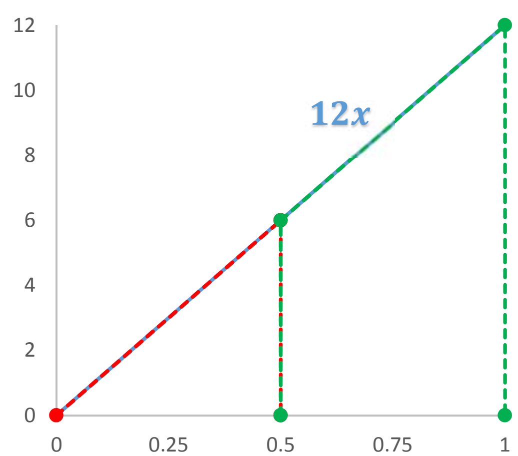
$$I_i exact = \int_0^1 12x\; dx = 6x^2 = 6.0000$$

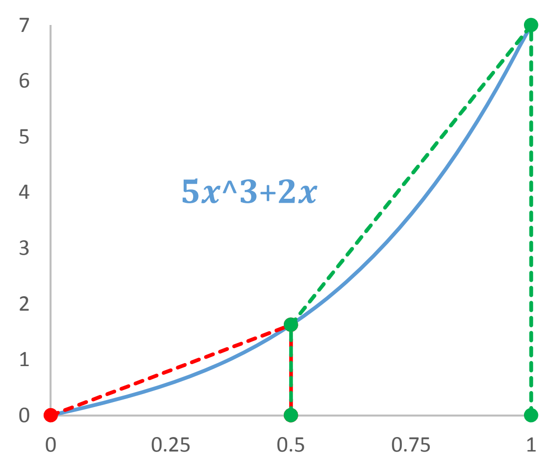
$$e_i = 6.0000 - 6.0000 = 0.0000$$

$$I_{ii} \approx 0.5 * \left[\frac{1}{2}f(0) + f(0.5) + \frac{1}{2}f(1)\right] = 2.5625\,, h = \frac{b-a}{2} = 0.5, n = 2$$

$$I_{ii} exact = \int_0^1 5x^3 + 2x\; dx = \frac{5}{4}x^4 + x^2 = \frac{9}{4} = 2.2500$$

$$e_{ii} = 2.5625 - 2.2500 = 0.3125$$





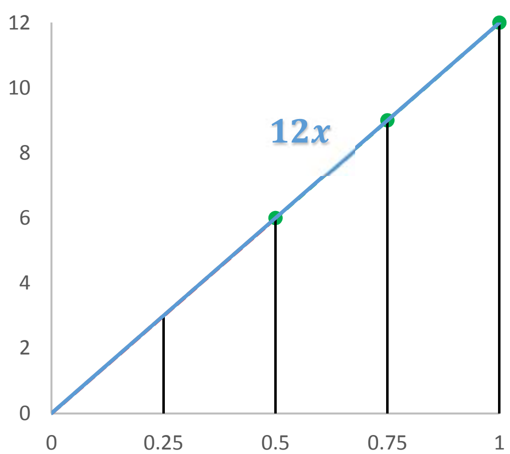
$$I_{Simpson} = \int_0^1 f(x)dx \approx \frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + f_4] \ , h = \frac{b-a}{2n}$$

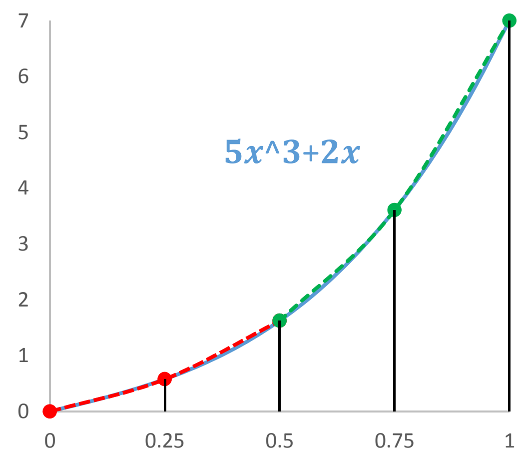
$$I_i \approx \frac{h}{3}[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)] = 6.00 \ , h = \frac{1-0}{4} = 0.25, n = 2$$

$$e_i = 6.00 - 6.00 = 0.00$$

$$I_i \approx \frac{h}{3}[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)] = 2.25 \ , h = \frac{1-0}{4} = 0.25, n = 2$$

$$e_i = 2.25 - 2.25 = 0.00$$





- ❖ Both methods are behaving as expected, since the first function is linear both Trapezoidal and Simpson would act exactly but for the second function which is from the third order, linear integration (Trapezoidal) would cause errors, which was calculated.

10*. Perform the numerical integration of

$$\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y)\, dx\, dy$$

using Simpson's rule in each direction. Is the approximation behaving as expected?

Since the Double Integral is introducing the volume under a surface and considering the fact that the function is completely dividable to 2 functions any of which corresponding to one variable, then we have:

$$\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y)\, dx\, dy = \int_0^1 (9x^3 + 8x^2)\, dx * \int_0^1 (y^3 + y)\, dy$$

$$= \left(\frac{9}{4}x^4 + \frac{8}{3}x^3\right) * \left(\frac{1}{4}y^4 + \frac{1}{2}y^2\right) = \left(\frac{9}{4} + \frac{8}{3}\right) * \left(\frac{1}{4} + \frac{1}{2}\right) = 4.9167 * 0.75 = 3.6875$$

a	0
b	1
n	1
h	0.5

x	f(x)	Ψ	Ψ.f(x)
0	0	1	0
0.5	3.125	4	12.5
1	17	1	17
	ΣΨ.f(x)		29.5
	h/3(ΣΨ.f(x))		4.91666667

y	f(y)	Ψ	Ψ.f(y)
0	0	1	0
0.5	0.625	4	2.5
1	2	1	2
	ΣΨ.f(y)		4.5
	h/3(ΣΨ.f(y))		0.75

[h/3(ΣΨ.f(x))]*[h/3(ΣΨ.f(y))]	3.6875

Due to the nature of functions, Simpson Integration is well suited to this case whitout the need to have further subintervals. But we could check this fact by having for instance 5 sections (n=5) and comparing the results by the first attempt (n=1).

a	0
b	1
n	5
h	0.1

x	f(x)	Ψ	Ψ.f(x)
0	0	1	0
0.1	0.089	4	0.356
0.2	0.392	2	0.784
0.3	0.963	4	3.852
0.4	1.856	2	3.712
0.5	3.125	4	12.5
0.6	4.824	2	9.648
0.7	7.007	4	28.028
0.8	9.728	2	19.456
0.9	13.041	4	52.164
1	17	1	17

ΣΨ.f(x)	147.5
h/3(ΣΨ.f(x))	4.91666667

y	f(y)	Ψ	Ψ.f(y)
0	0	1	0
0.1	0.101	4	0.404
0.2	0.208	2	0.416
0.3	0.327	4	1.308
0.4	0.464	2	0.928
0.5	0.625	4	2.5
0.6	0.816	2	1.632
0.7	1.043	4	4.172
0.8	1.312	2	2.624
0.9	1.629	4	6.516
1	2	1	2

ΣΨ.f(y)	22.5
h/3(ΣΨ.f(y))	0.75

[h/3(ΣΨ.f(x))]*[h/3(ΣΨ.f(y))]	3.6875