NM4PDEs - Exercises ODEs

1. The motion of a non-frictional pendulum is governed by the Ordinary Differential Equation (ODE)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where θ is the angular displacement, $L = 1$ m is the pendulum length and the gravity acceleration is $g = 9.8$ m/s^2.

The position and velocity at time $t = 1$ s are known:

$$\theta(1) = 0.4 \text{ rad} \quad ; \quad \frac{\mathrm{d}\theta}{\mathrm{dt}}(1) = 0 \text{ rad/s}$$

a) Solve the initial boundary value problem in the interval $(0, 1)$ using a second-order Runge-Kutta method to determine the initial position at $t = 0$ s, with 2 and 4 time steps.

b) Using the approximations obtained in *a)*, compute an approximation of the relative error in the solution computed with 2 steps.

c) Propose a time step h to obtain an approximation with a relative error three orders of magnitude smaller.

a) initial position $\mathrm{t} = 0$ *with Runge-Kutta with 2 and 4 steps*

Using Heun's method

$$\boldsymbol{y}^{i+1} = \boldsymbol{y}^i + \frac{h}{2}(k_1 + k_2)$$

$$\boldsymbol{k_1} = f(x^i, \boldsymbol{y}^i)$$

$$\boldsymbol{k_2} = f(x^i + h, \boldsymbol{y}^i + k_1 h)$$

Applied to a system of two equations equivalent to the second order differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$$y_1 = \theta$$

$$y_2 = \frac{d\theta}{dt}$$

This change of variables yields the system

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -g/L & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

with

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\boldsymbol{f}(x, \boldsymbol{y}) = \begin{bmatrix} 0 & 1 \\ -g/L & 0 \end{bmatrix} \boldsymbol{y}^i$$

Applying Heun's method with two steps $h = -0.5\text{s}$

- Step 1

$$\boldsymbol{k_1} = \begin{bmatrix} 0 & 1 \\ -9.8 & 0 \end{bmatrix} \boldsymbol{y}^i$$

$$\boldsymbol{y}^{t=1} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$$

$$\boldsymbol{k_1} = \begin{bmatrix} 0 & 1 \\ -9.8 & 0 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3.92 \end{bmatrix}$$

$$\boldsymbol{k_2} = \begin{bmatrix} 0 & 1 \\ -9.8 & 0 \end{bmatrix} \left(\begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - 0.5 \begin{bmatrix} 0 \\ -3.92 \end{bmatrix} \right) = \begin{bmatrix} 1.96 \\ -3.92 \end{bmatrix}$$

$$\boldsymbol{y}^{t=0.5} = \boldsymbol{y}^{t=1} - \frac{h}{2}(\boldsymbol{k_1} + \boldsymbol{k_2}) = \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix}$$

- Step 2

$$\boldsymbol{y}^{t=0.5} = \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix}$$

$$\boldsymbol{k_1} = \begin{bmatrix} 0 & 1 \\ -9.8 & 0 \end{bmatrix} \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix} = \begin{bmatrix} 1.96 \\ 0.882 \end{bmatrix}$$

$$\boldsymbol{k_2} = \begin{bmatrix} 0 & 1 \\ -9.8 & 0 \end{bmatrix} \left(\begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix} - 0.5 \begin{bmatrix} 0 \\ -3.92 \end{bmatrix} \right) = \begin{bmatrix} 1.519 \\ 10.486 \end{bmatrix}$$

$$\boldsymbol{y}^{t=0} = \boldsymbol{y}^{t=0.5} - \frac{h}{2}(\boldsymbol{k_1} + \boldsymbol{k_2}) = \begin{bmatrix} -0.9598 \\ -0.8820 \end{bmatrix}$$

Applying Heun's method with four steps $h = -0.25\text{s}$

Step	$\boldsymbol{y}^i$	$\boldsymbol{k_1}$	$\boldsymbol{k_2}$	$\boldsymbol{y}^{i+1}$
1	$\begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -3.92 \end{bmatrix}$	$\begin{bmatrix} 0.98 \\ -3.92 \end{bmatrix}$	$\begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix}$
2	$\begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix}$	$\begin{bmatrix} 0.98 \\ -2.7195 \end{bmatrix}$	$\begin{bmatrix} 1.6599 \\ -0.3185 \end{bmatrix}$	$\begin{bmatrix} -0.0525 \\ 1.3598 \end{bmatrix}$
3	$\begin{bmatrix} -0.0525 \\ 1.3598 \end{bmatrix}$	$\begin{bmatrix} 1.3598 \\ 0.5143 \end{bmatrix}$	$\begin{bmatrix} 1.2312 \\ 3.8457 \end{bmatrix}$	$\begin{bmatrix} -0.3763 \\ 0.8147 \end{bmatrix}$
4	$\begin{bmatrix} -0.3763 \\ 0.8147 \end{bmatrix}$	$\begin{bmatrix} 0.8147 \\ 3.6882 \end{bmatrix}$	$\begin{bmatrix} -0.1073 \\ 5.6843 \end{bmatrix}$	$\begin{bmatrix} -0.4648 \\ -0.3568 \end{bmatrix}$

b) Relative error

$$y_1^a(t) = A \ \sin(\omega t + \emptyset)$$

$$d_t y_1^a(t) = y_2^a(t) = A\omega \ \ \cos(\omega t + \emptyset)$$

$$d_{tt} y_1^a(t) = -A\omega^2 \ \ \sin(\omega t + \emptyset)$$

$$\omega^2 = g/L$$

$$y_2^a(t=1) = A\omega \ \ \cos\left(\sqrt{g/L} + \emptyset\right) = 0$$

$$\emptyset = \left(n + \frac{1}{2}\right)\pi - \sqrt{g/L} \qquad\qquad n = 0,1,2,\dots$$

$$y_1^a(t=1) = A \ \ \sin\left(\sqrt{g/L} + \emptyset\right) = 0.4$$

$$A = 0.4$$

$$y_1^a(t=0) = 0.4 \ \ \sin\left(\left(n + \frac{1}{2}\right)\pi - \sqrt{g/L}\right) = -0.399975$$

$$\epsilon = \left|\frac{y_1 - y_1^a}{y_1^a}\right| = \left|\frac{-0.9598 + 0.399975}{-0.399975}\right| \approx 140\%$$

c) Time step h for error three orders of magnitude smaller

Number of steps	Error ϵ, %
2	140%
4	16%
8	2.2%
10	1.1%

2. Consider the initial value problem

$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0,1)$$
$$y(0) = 1$$

a) Solve the initial value problem using the Euler method with step $h = 0.25$.

b) Compute the solution using the Heun method with a step h such that the computational cost is equivalent to the computational cost in *a)*.

Note that the analytical solution of the initial value problem is a second degree polynomial.

a) Solve usgin Euler

$$Y_{i+1} = Y_i + hf(x_i, Y_i)$$

Step i	x_i	Y_i	$f(x_i, Y_i)$	Y_{i+1}
0	0	1	2	1.5
1	0.25	1.5	2.438	2.109
2	0.5	2.109	2.859	2.824
3	0.75	2.824	3.262	3.640
4	1	3.640	3.640	4.550

a) *Compute using Heun with equivalent computational cost as in a).*

The computational cost in question *a)* is that of the evaluation of $f(x_i, Y_i)$ for 4 times.

The Heun method evaluates $f(x_i, Y_i)$ 4 times per time step. Therefore the equation must be integrated using the Heun method in one signle time step.

$$Y_{i+1} = Y_i + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = f(x_i, Y_i)$$

$$k_2 = f\left(x_i + \frac{h}{2}, Y_i + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(x_i + \frac{h}{2}, Y_i + \frac{h}{2}k_2\right)$$

$$k_4 = f\left(x_i + \frac{h}{2}, Y_i + hk_3\right)$$

Step i	x_i	Y_i	k_1	k_2	k_3	k_4	Y_{i+1}
0	0	1	2	2.75	3.125	4.125	3.979

$$Y_1 = 3.979$$

Analytically

$$y = ax^2 + bx + c$$

$$\frac{dy}{dx} = 2ax + b$$

Substituting this two in the equation

$$2ax + b = ax^2 + bx + c - x^2 + 1$$

$$a = 1 \quad ; \quad b = 2 \quad ; \quad c = 1$$

$$y(x = 1) = ax^2 + bx + c = 4$$

The Heun method acheives in one step a global error of 0.5% whereas Euler method acheives 14% in 4 time steps.

3. The ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

is defined over the domain (0,1), and is to be solved numerically subject to the initial condition $y(0) = 1$, where $y(x)$ is the exact solution. The forward Euler method for integrating the above differential equation is written as

$$Y_{i+1} = Y_i + hf(x_i, Y_i)$$

where Y_i denotes the discrete solution at node i, with position x_i, of a uniform grid of nodes of constant grid interval size h and $x_{i+1} = x_i + h$.

a) Using a Taylor series expansion, deduce the leading truncation error of the scheme. Is the method consistent? Explain your answer.

b) State the backward Euler method for integrating the above differential equation where $f(x, y)$ is a general non-linear function of x and y.

c) Deduce the stability limits of the respective forward Euler method and backward Euler method for the model equation $dy/dx = -\lambda y$ where λ is a positive real constant.

d) Use the backward Euler method to compute the numerical solution of the ordinary differential equation

$$\frac{dy}{dx} = -25y^{3.5}$$

with initial condition $y(0) = 1$, by hand for two steps with grid interval size $h = 1/10$. (Use 2 Newton iterations per step for this calculation.)

e) Use the forward Euler method to compute the numerical solution of the above ordinary differential equation with same initial condition by hand for two steps with grid interval size h=1/10.

f) The analytical solution is

$$y(x) = \left(\frac{125x + 2}{2}\right)^{-2/5}$$

Using Matlab codes, indicate the maximum stable interval size possible for forward Euler method from the following; h=1/10, h=1/15, h=1/30, h=1/45, h=1/90. How does your choice compare with the stability condition?

a) Deduce truncation error for forward Euler

The Taylor series expansion reads

$$y(x) = y(x_0) + \frac{x - x_0}{1!}\frac{dy}{dx}(x_0) + \cdots$$

For a finite step of x, h

$$y_{i+1} = y_i + h\frac{dy}{dx}(x_i) + \mathcal{O}(h^2)$$

$$\frac{dy}{dx}(x_i) = \frac{y_{i+1} - y_i}{h} - \tau_i(h)$$

$$\tau_i(h) = \frac{\mathcal{O}(h^2)}{h} = \mathcal{O}(h)$$

The method is consistent because $\tau_i(h) = \mathcal{O}(h) \longrightarrow 0$ when $h \longrightarrow 0$

b) State backward Euler to integrate the differential equation where f(x,y) is a general non-linear function

The backward Euler method is an implicit method

$$Y_{i+1} = Y_i + hf(x_{i+1}, Y_{i+1}) \qquad i = 0, \dots m-1$$

$$Y_0 = 1$$

Given a step h to increment x_i, and an initial condition at x_0, $Y(0) = Y_0$, the method proceeds for every step i

1. Compute $x_{i+1} = x_i + h$
2. Solve for Y_{i+1} the implicit equation $0 = Y_i + hf(x_{i+1}, Y_{i+1}) - Y_{i+1} = g(Y_{i+1})$
   a. Guess a solution for Y_{i+1}, for instance $Y_{i+1}^0 = Y_i$
   b. Compute the gradient $\frac{dg}{dY}(Y_{i+1}^0)$
   c. Compute $Y_{i+1}^{j+1} = Y_{i+1}^{j} - \left[\frac{dg}{dY}(Y_{i+1}^0)\right]^{-1} g(Y_{i+1})$
   d. Check convergence $Y_{i+1}^{j+1} - Y_{i+1}^{j} < \varepsilon$

c) Deduce stability limits for forward Euler and backward Euler methods for the model equation $dy/dx = -\lambda y$ where λ is a positive real constant.

Forward Euler

$$Y_{i+1} = Y_i - h\lambda Y_i$$

$$Y_{i+1} = GY_i \qquad \text{with} \qquad G = 1 - h\lambda$$

The scheme is absolutely stable if $|G| < 1$

Therefore, the forward Euler method applied to the model equation is stable for

$$\left.\begin{array}{lcl} 1 - h\lambda < 1 & \& & 1 - h\lambda > 0 \\ -1 + h\lambda < 1 & \& & 1 - h\lambda < 0 \end{array}\right\}$$

The method is conditionally stable because there are possible values of λ that breach this condition

$$(0 < h\lambda < 1) \cup (1 < h\lambda < 2)$$

d) Numerical solution with backward Euler

$$\frac{dy}{dx} = -25y^{3.5}$$

$$y(0) = 1$$

$$\mathrm{h} = 1/10$$

Given a step h to increment x_i, and an initial condition at x_0, $Y(0) = Y_0$, the method proceeds for every step i

1. Compute $x_{i+1} = x_i + h$
2. Solve for Y_{i+1} the implicit equation $0 = Y_i + hf(x_{i+1}, Y_{i+1}) - Y_{i+1} = g(Y_{i+1})$
   a. Guess a solution for Y_{i+1}, for instance $Y_{i+1}^0 = Y_i$
   b. Compute the gradient $\frac{dg}{dY_{i+1}}(Y_{i+1}^0)$
   c. Compute $Y_{i+1}^{j+1} = Y_{i+1}^j - \left[\frac{dg}{dY}(Y_{i+1}^0)\right]^{-1} g(Y_{i+1})$
   d. Check convergence $Y_{i+1}^{j+1} - Y_{i+1}^j < \varepsilon$

Step	x_i	Y_i	iter	Y_{i+1}^j	$g(Y_{i+1})$	$\left[\frac{dg}{dY}(Y_{i+1}^0)\right]$	Y_{i+1}^{j+1}
0	0	1	0	1	-2.5	-9.75	0.7436
			1	0.7436	-0.6299	-5.1720	0.6218
			2	0.6218	-0.0957	-3.6676	**0.5957**
1	0.1	0.5957	0	0.5957	-0.4079	-3.3965	0.4756
			1	0.4756	-0.0654	-2.3650	0.4480
			2	0.4480	-0.0027	-2.1752	**0.4467**

e) Numerical solution with forward Euler

$$Y_{i+1} = Y_i + hf(x_i, Y_i)$$

$$Y_{i+1} = Y_i - 25hY_i^{3.5}$$

Step	x_i	Y_i	Y_{i+1}
0	0	1	-1.5
1	0.1	2	NA

The square root of a negative real yields a Not a Number error. Apparently, this method has proofed not to be stable for this equation

f) Maximum stable interval size

The method is stable only for intervals $\mathrm{h} = 1/30$ and smaller.

4. The second-order ordinary differential equation

$$\frac{d^2y}{dx} + \omega^2 y = 0$$

is defined over the domain $(0, 1)$, and is to be solved numerically subject to the initial conditions $y(0) = 0$, $dy/dx(0) = \omega$, where $y(x)$ is the exact solution.

a) Reduce the above second order ODE to a system of first order ODEs.

b) Set $\omega^2 = 3$. Using the forward Euler method to integrate the system, compute the solution at $t = 1$ by hand with $n = 4$ steps. Use the Forward Euler code to check your results.

c) Using the Matlab code, compute the solution using $n = 8$ steps. Use these solution values to estimate the step size required to obtain a numerical solution with three significative digits. Try your new step size.

a) Reduce to first order ODEs

$$y_1' = y_2$$

$$y_2' = -\omega^2 y_1$$

With initial conditions

$$y_1(0) = 0$$

$$y_2(0) = \omega$$

b) set $\omega^2 = 3$ *and compute using forward Euler*

$$\mathbf{Y}_{i+1} = \mathbf{Y}_i + h\mathbf{f}(\mathbf{x}_i, \mathbf{Y}_i)$$

Step	$Y_{1,i}$	$Y_{2,i}$	$Y_{1,i+1}$	$Y_{2,i+1}$
0	0	1.7321	0.4330	1.7321
1	0.4330	1.7321	0.8660	1.4073
2	0.8660	1.4073	1.2178	0.7578
3	1.2178	0.7578	**1.4073**	**-0.1556**

c) compute using forward Euler with 8 steps and estimate the step size required to obtain a numerical solution with three significant digits

$$\mathbf{Y}_{i+1} = \mathbf{Y}_i + h\mathbf{f}(\mathbf{x}_i, \mathbf{Y}_i)$$

Step	$Y_{1,i}$	$Y_{2,i}$	$Y_{1,i+1}$	$Y_{2,i+1}$
0	0.0000	1.7321	0.2165	1.7321
1	0.2165	1.7321	0.4330	1.6509
2	0.4330	1.6509	0.6394	1.4885
3	0.6394	1.4885	0.8254	1.2487
4	0.8254	1.2487	0.9815	0.9392

5	0.9815	0.9392	1.0989	0.5711
6	1.0989	0.5711	1.1703	0.1590
7	1.1703	0.1590	**1.1902**	**-0.2798**

The Global error is

$$E = Ch^{p+1}$$

For the 4 steps and 8 steps methods

$$E_4 = Ch_4^{p+1}$$

$$E^* = C{h^*}^{p+1} = Tol$$

$$h^* = \left(\frac{Tol}{E_4}\right)^{1/(p+1)}$$

The error made with 4 steps can be estimated by comparing the results with 4 steps with the results obtained with 8 steps. Then,

$$h^* = \left(\frac{0.0001}{0.2798 - 0.1556}\right)^{1/2} = 0.028$$

A step size $h = 0.025$ to make it a divisor of the total time $t = 1$.

Using this step size for 40 time steps the following table of results is obtained.

Step	$Y_{1,i}$	$Y_{2,i}$	$Y_{1,i+1}$	$Y_{2,i+1}$
0	0.0000	1.7321	0.0433	1.7321
1	0.0433	1.7321	0.0866	1.7288
2	0.0866	1.7288	0.1298	1.7223
3	0.1298	1.7223	0.1729	1.7126
…				
36	1.0342	0.0232	1.0348	-0.0544
37	1.0348	-0.0544	1.0334	-0.1320
38	1.0334	-0.1320	1.0301	-0.2095
39	1.0301	-0.2095	**1.0249**	**-0.2868**