Soskic Laosuelelt Tomas
Ex 3. PDEs

2.

$u_t + a u_x = 0$ for $x \in (0,1)$, $t \geq 0$ and $a > 0$

$u(x,0) = \sin(2\pi x)$

$u(0,t) = u(1,t)$

a)

Using a first order upwind implicit scheme (Backward in time and forward in space) the PDE can be written as:

$$\left(1 + a\frac{\Delta t}{\Delta x}\right) u_i^{n+1} - a\frac{\Delta t}{\Delta x} u_{i-1}^{n+1} = u_i^{n-1} \quad \text{for } i = 1, \dots, M$$

b) Treatment of BC

For the boundary at $i = 0$

$$\left(1 + a\frac{\Delta t}{\Delta x}\right) u_0^{n+1} - a\frac{\Delta t}{\Delta x} u_{-1}^{n+1} = u_0^n$$

Since the boundary conditions are periodic: $u_{-1} = u_M$, therefore the system to be solved is

$$\begin{pmatrix} 1 + a\frac{\Delta t}{\Delta x} & 0 & \cdots & \cdots & -a\frac{\Delta t}{\Delta x} \\ -a\frac{\Delta t}{\Delta x} & 1 + a\frac{\Delta t}{\Delta x} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & -a\frac{\Delta t}{\Delta x} & 1 + a\frac{\Delta t}{\Delta x} \end{pmatrix} \begin{pmatrix} u_0^{n+1} \\ \vdots \\ \vdots \\ u_M^{n+1} \end{pmatrix} = \begin{pmatrix} u_0^n \\ \\ \\ u_M^n \end{pmatrix}$$

c)

Direct Method:

Iterative Method: Gauss-Seidel because the matrix is diagonally dominant and the convergence is faster than in Jacobi

4-

$u_t = \nu u_{xx} + \sigma u$ for $x \in (0,1)$ and $t > 0$

BC: $\begin{cases} u(0,t) = 0 \\ \dfrac{\partial u}{\partial x}(1,t) = 0 \end{cases}$

Initial conditions : $u(x,0) = \underbrace{\begin{cases} 0 & x < 1/4 \\ 4x-1 & \frac{1}{4} \leq x < \frac{1}{2} \\ -4x+3 & \frac{1}{2} \leq x < 3/4 \\ 0 & \frac{3}{4} \leq x \end{cases}}_{f(x)}$

a) Explicit scheme

- Using FTCS:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\nu (u_{i-1}^n - 2u_i^n + u_{i+1}^n)}{\Delta x^2} + \sigma u_i^n,$$

$$u_i^{n+1} = \frac{\nu \Delta t}{\Delta x^2} u_{i-1}^n + \left(1 + \sigma \Delta t - \frac{2\Delta t \nu}{\Delta x^2}\right) u_i^n + \frac{\nu \Delta t}{\Delta x^2} u_{i+1}^n$$

- Dirichlet BC $\Rightarrow u_0^{n+1} = 0$

- ~~von~~ Neumann BC $\Rightarrow$ Computing the solution at $i = M+1$:

$$u_{M+1}^{n+1} = \frac{\nu \Delta t}{\Delta x^2} u_M^n + \left(1 + \sigma \Delta t - \frac{2\Delta t \nu}{\Delta x^2}\right) u_{M+1}^n + \frac{\nu \Delta t}{\Delta x^2} u_{M+2}^n$$

u_{M+2}^n can be computed approximating the derivative $\left.\dfrac{\partial u}{\partial x}\right|_{M+1} = \dfrac{u_{M+2}^n - u_M^n}{2\Delta x} = 0$ at the boundary

$\Rightarrow u_{M+2}^n = u_M^n \rightsquigarrow$ Substituting at the solution at $i = M+1$

$$u_{M+1}^{n+1} = \frac{2\nu \Delta t}{\Delta x^2} u_M^n + \left(1 + \sigma \Delta t - \frac{2\Delta t \nu}{\Delta x^2}\right) u_{M+1}^n$$

- Initial conditions

$$u_i^0 = \begin{cases} 0 & \text{for } x < 1/4 \\ 4x-1 & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\ -4x-3 & \frac{1}{2} \leq x < \frac{3}{4} \\ 0 & \frac{3}{4} \leq x \end{cases}$$

b)

i) $\sigma = 0$

- $u_i^{n+1} = \nu \dfrac{\Delta t}{\Delta x^2} u_{i-1}^n + \left(1 + \cancel{\sigma \Delta t} - \dfrac{2\Delta t \nu}{\Delta x^2}\right) u_i^n + \dfrac{\nu \Delta t}{\Delta x^2} u_{i+1}^n$
- Dirichlet: $u_0^{n+1} = 0$
- Neumann : $u_{M+1}^{n+1} = \dfrac{2\nu \Delta t}{\Delta x^2} + \left(1 - \dfrac{2\Delta t \nu}{\Delta x^2}\right) u_{M+1}^n$

And the same initial conditions

ii) $\nu = 0$

- $u_i^{n+1} = (1 + \sigma \Delta t) u_i^n$
- Dirichlet: $u_0^{n+1} = 0$
- Neumann : $u_{M+1}^{n+1} = (1 + \sigma \Delta t) u_{M+1}^n$

And the same initial conditions

4-

c)

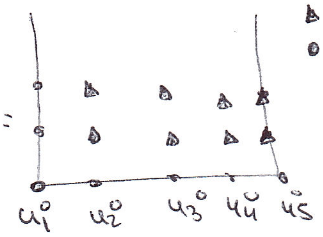
$\nu = 0,1$

$\sigma = -0,1$

$\Delta t = 0,1$

$\Delta x = 0,25$

$\left.\right\} \frac{\nu \Delta t}{\Delta x} = 0,16$, $\left(1+\sigma\Delta t - \frac{2\nu\Delta t}{\Delta x}\right) = 0,67$



▲ unknowns

● Known through BC and initial conditions

- Initial conditions

$u_1^0 = 0$, $u_2^0 = 0$, $u_3^0 = 1$, $u_4^0 = 0$, $u_5^0 = 0$

- $t = 0,1$

$u_1^1 = 0$, $u_2^1 = 0,16$, $u_3^1 = 0,67$, $u_4^1 = 0,16$, $u_5^1 = 0$

- $t = 0,2$

$u_1^2 = 0$, $u_2^2 = 0,2144$, $u_3^2 = 0,5001$, $u_4^2 = 0,2144$, $u_5^2 = 0,0512$

As it can be seen in Fig 1 (at the end) the diffusion effects ~~cause the profile of u~~ cause a widening of the profile

d) Implicit scheme

- Using BTCS

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right)}{\Delta x^2} + \sigma u_i^{n+1} ,$$

$$-\frac{\nu\Delta t}{\Delta x^2} u_{i-1}^{n+1} + \left(1 - \sigma\Delta t + \frac{2\Delta t \nu}{\Delta x^2}\right) u_i^{n+1} - \frac{\nu\Delta t}{\Delta x^2} u_{i+1}^{n+1} = u_i^n$$

- Dirichlet BC

$u_0^{n+1} = 0$

- Neumann BC

As in the explicit scheme $u_{M+2}^n = u_M^n$ so that:

$$-\frac{2\nu\Delta t}{\Delta x^2} u_M^{n+1} + \left(1 + \frac{2\nu\Delta t}{\Delta x^2} - \sigma\Delta t\right) u_{M+1}^{n+1} = u_{M+1}^n$$

And the matrix can be written as

$$M = \begin{pmatrix} 1 + \frac{2\nu\Delta t}{\Delta x^2} - \sigma\Delta t & -\frac{\nu\Delta t}{\Delta x^2} & \cdots & & 0 \\ -\frac{\nu\Delta t}{\Delta x^2} & 1 - \sigma\Delta t + \frac{2\nu\Delta t}{\Delta x^2} & -\frac{\nu\Delta t}{\Delta x^2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \cdots & -\frac{2\nu\Delta t}{\Delta x^2} & 1 + \frac{2\nu\Delta t}{\Delta x^2} - \sigma\Delta t \end{pmatrix}$$

which can be solved through Gaussian elimination for tridiagonal matrices

Fig 1.