Navier-Stokes Element Functional using Enriched Pressure

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January 9, 2017

Abstract

The objective of this work is to present the formulation of a functional for a Navier Stokes element created to be implemented in a level-set solver capable of dealing with two fluids by tracking the interface between the two. In order to fulfill this objective it is necessary to use a space of interpolation of the higher pressure of usual by adding an enrichment.

1 Introduction

During the last decades different methods have been proposed to deal with problems involving fluids in different phases, typically water and air. A basic approach to this problem is to ignore the air as the effect of water is always dominant. However, there are certain problems like sloshing or foaming in which air can not be neglected. Within the Eulerian methods, one that has achieved great notoriety is the level-set method that allows to follow where the interface between the fluids is.

In this work, the formulation of a Navier-Stokes element that can be implemented in a level-set is exposed in some detail. The objective is to eliminate a formulation of the stiffness matrix and the force vector for a demarcated gauss point and that can be directly assembled in the global system. For this we will start from the Navier-Stokes equations to which we must make a modification enriching the pressure. The reason for this is the incapacity of the usual pressure interpolation to capture the discontinuity of the gradient of the pressure on the interface. As this gradient and the velocity are coupled in the momentum equation of the Navier Stokes equations, this will lead to spurious velococites near the interface that will show unrealistic solutions even for the simplest problems. This issue will be overcome by adding a new set of shape functions that are continuous but have discontinuous gradients enriching the pressure interpolation on the interface. In this section it will be explained how to apply the Algebraic Subgrid Scales Stabilization to this problem (subsection 2.1) and how to obtain the corresponding functional (subsection 2.3)

2 Methodology

2.1 Governing Equations

The Navier Stokes equations for an incompressible fluid in a domain Ω are:

$$\begin{aligned} \rho\partial_t\mathbf{u} + \rho\mathbf{a}\cdot\nabla\mathbf{u} + \nabla p - \nabla\cdot\mathbb{C}\nabla^s\mathbf{u} = \rho\mathbf{f} \qquad & in\ \Omega\times(0,T) \\ \rho(\nabla\cdot\mathbf{u}) = 0 \qquad & in\ \Omega\times(0,T) \end{aligned} \tag{1}$$

Where ρ is the density of the fluid, which will be considered constant, $\mathbf{u}$ is the unkown velocity, $\mathbf{a}$ is the convection velocity, p is the pressure, $\mathbb{C}$ is the constitutive matrix and $\mathbf{f}$ represent the body forces.

The problem must be completed with suitable initial and boundary conditions. We will split the boundary of the domain $\partial\Omega$ into the Dirichlet boundary Γ_D and the Neumann boundary Γ_N, hence $\partial\Omega = \Gamma_D \cup \Gamma_N$.

$$\begin{aligned} \mathbf{u}(\mathbf{x},0) = \mathbf{u_0}(\mathbf{x}) \qquad & in\ \Omega, t=0 \\ \mathbf{u}(\mathbf{x},t) = \mathbf{u_D}(\mathbf{x},t) \qquad & on\ \Gamma_D\times(0,T) \\ \mathbf{n}\cdot\boldsymbol{\sigma}(\mathbf{x},t) = \ \mathbf{t}(\mathbf{x},t) \qquad & on\ \Gamma_N\times(0,T) \end{aligned} \tag{2}$$

Where u_0 corresponds to the initial velocity field, u_D represents the imposed velocity on the Dirichlet boundary, $\boldsymbol{n}$ is the outer normal vector, t is the imposed traction on the Neumann boundary and $\boldsymbol{\sigma}$ is the stress tensor which can be related to the unknowns $\boldsymbol{u}$ and p with:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \mathbb{C}\nabla^s\boldsymbol{u} \tag{3}$$

The corresponding residuals are:

$$\mathbf{r_m}(\mathbf{u},p) = \rho\mathbf{f} - \rho\partial_t\mathbf{u} - \rho\mathbf{a}\cdot\nabla\mathbf{u} - \nabla p + \nabla\cdot\mathbb{C}\nabla^s\mathbf{u} \tag{4}$$

$$r_c(\mathbf{u}) = -\rho(\nabla\cdot\mathbf{u}) \tag{5}$$

We test against the functions $\mathbf{w}$ and q and integrate over the domain. The pressure and the viscous term will be integrated by parts obtaining the weak form of the problem. We will aso add an extra equation testing the continuity equation against q^*:

$$\begin{aligned} (\mathbf{w},\rho\partial_t\mathbf{u} + \rho\mathbf{a}\cdot\nabla\mathbf{u}) - (\nabla\cdot\mathbf{w}p) + (\nabla\mathbf{w} : \mathbb{C}\nabla^s\mathbf{u}) = (\mathbf{w},\rho\mathbf{f}) + (\boldsymbol{w},t)_{\Gamma_N} \qquad & \forall w\in\mathcal{V} \\ (q,\rho\nabla\cdot\mathbf{u}) = 0 \qquad & \forall q\in\mathcal{Q} \\ (q^*,\rho\nabla\cdot\mathbf{u}) = 0 \qquad & \forall q^*\in\mathcal{Q}^* \end{aligned} \tag{6}$$

Where $\mathcal{V} \in (\mathcal{H}^1)^d$, $\mathcal{Q} \in \mathcal{L}^2$ and $\mathcal{Q}^* \in \mathcal{H}^1$ represent the following functional spaces:

$$\mathcal{H}^1(\Omega)^d := \left\{ v : \Omega \to \mathbb{R}^d \mid \int_\Omega |v|^2 < \infty, \int_\Omega |\nabla v|^2 < \infty \right\} \tag{7}$$

$$\mathcal{L}^2(\Omega) := \left\{ v : \Omega \to \mathbb{R} \mid \int_\Omega v^2 < \infty \right\} \tag{8}$$

2.2 Algebraic Subgrid Scales Stabilization

In order to be able to use linear interpolations for both the velocity and the pressure we need to add stabilization terms to the Galerkin method. We will basically follow the same lines proposed in [1]. Let's consider a subscale decomposition as follows:

$$\mathbf{u} = \mathbf{u_h} + \mathbf{u}_s \tag{9}$$

$$p = p_h + p_s \tag{10}$$

The terms $\mathbf{u}_h$ and p_h represent the part of the solution which belongs to the finite element space $\mathbf{u_h} \in \mathcal{V}_h$ while $\mathbf{u}_s$ and p_s belong to the small scale space such that $\mathcal{V} = \mathcal{V}_h \oplus V_s$ and $\mathcal{Q} = \mathcal{Q}_h \oplus \mathcal{Q}_s$.

As we mentioned in the introduction we will use a enriched pressure field as follows:

$$p_h^{tot} = \sum_i^{nnodes} N_i p_i + \sum_j^{nnodes} N_j^* p_j^* \tag{11}$$

Where N corresponds here to the usual finite elements shape functions while N^* are the enrichment functions. We will have analogously a new set of unknowns p^* for our pressure field. We can conclude that $p_h^{tot} \in \mathcal{Q}_h \cup \mathcal{Q}_h^*$

The enricment shape functions N^* will be constructed in the following way for an element cut by the interface Γ: for a node i the element can be splitted into two parts separated by the interface Γ, one subdomain Ω_e^i which contains the node i and other subdomain Ω_e^{-i} which does not include it. The enriched shape function N_i^* for node i will be equal to zero in the subdomain Ω_e^i and equal to N_i in the subdomain Ω_e^{-i} .

$$\begin{aligned} N_i^* &= N_i \qquad in\ \Omega_e^{-i} \\ N_i^* &= 0 \qquad in\ \Omega_e^{i} \end{aligned} \tag{12}$$

Using a quasi-static small scales model where $\partial_t u_s = 0$ and:

$$\mathbf{u}_s \approx \tau_1 \mathbf{r_m}(\mathbf{u_h}, p_h^{tot}) \tag{13}$$

$$p_s \approx \tau_2 r_c(\mathbf{u_h}) \tag{14}$$

we obtain the following weak form with the added stabilization terms.

$$(\mathbf{w}_h, \rho\partial_t \mathbf{u_h} + \rho\mathbf{a}\cdot\nabla\mathbf{u_h}) - (\nabla\cdot\mathbf{w_h}p_h) + (\mathbf{w}_h, \nabla p_h^*) + (\nabla\mathbf{w}_h : \mathbb{C}\nabla^s\mathbf{u_h})$$
$$-\sum_{\Omega_e}\int_{\Omega_e}\rho(\mathbf{a}\cdot\nabla\mathbf{w}_h)\tau_1\cdot\mathbf{r_m}(\mathbf{u_h}, p_h^{tot}) - \sum_{\Omega_e}\int_{\Omega_e}\rho(\nabla\cdot\mathbf{w}_h)\tau_2 r_c(\mathbf{u_h}) = (\mathbf{w}_h, \rho\mathbf{f}) + (\boldsymbol{w}, t)_{\Gamma_N} \tag{15}$$

$$(q_h, \rho\nabla\cdot\mathbf{u_h}) = \sum_{\Omega_e}\int_{\Omega_e}\rho\nabla q_h\tau_1\cdot\mathbf{r_m}(\mathbf{u_h}, p_h^{tot}) \tag{16}$$

$$(q_h^*, \rho\nabla\cdot\mathbf{u_h}) = \sum_{\Omega_e}\int_{\Omega_e}\rho\nabla q_h^*\tau_1\cdot\mathbf{r_m}(\mathbf{u_h}, p_h^{tot}) \tag{17}$$

Notice that as we will use linear elements, the viscous term will vanish in $\mathbf{r_m}$ as it involves second spatial derivatives.

The terms τ_1 and τ_2 are the stabilization parameters which will be taken as:

$$\begin{aligned}\tau_1 =& \mathbf{I}\left(\frac{c_1\mu}{h^2} + \frac{c_2\rho|\boldsymbol{a}|}{h}\right)^{-1}\\ \tau_2 =& \mu + \frac{c_2|\boldsymbol{a}|h}{c_1}\end{aligned} \tag{18}$$

Where h is the characteristic length of the element and c_1 and c_2 are two constants that we have to define for each problem. For linear elements it is oftenly used $c_1 = 4$ and $c_2 = 2$

2.3 Functional with enriched pressure

The functional for the Navier Stokes problem with Subgrid Scales Stabilization can be constructed in the following way:

$$f(\mathbf{u_h}, p_h, p_h^*, \mathbf{w_h}, q_h, q_h^*) = f_g(\mathbf{u_h}, p_h, p_h^*, \mathbf{w_h}, q_h, q_h^*) + f_{stab}(\mathbf{u_h}, p_h, p_h^*, \mathbf{w_h}, q_h, q_h^*) \tag{19}$$

$$\begin{aligned}f_g(\mathbf{u_h}, p_h, p^*, \mathbf{w_h}, q_h, q_h^*) = {}& (\mathbf{w_h}, f) - \rho(\mathbf{w_h}, \partial_t\mathbf{u_h} + \mathbf{a}\cdot\nabla\mathbf{u_h}) - (\nabla\mathbf{w_h} : \mathbb{C}\nabla^s\mathbf{u_h}) - \\ & + (\nabla\cdot\mathbf{w_h}p_h) - (\mathbf{w_h}, \nabla p_h^*) - (q_h, \rho(\nabla\cdot\mathbf{u_h})) - (q_h^*, \rho(\nabla\cdot\mathbf{u_h}))\end{aligned} \tag{20}$$

$$f_{stab}(\mathbf{u_h}, p_h, p^*, \mathbf{w_h}, q_h, q_h^*) = \int_{\Omega_e} \rho(a \cdot \nabla \mathbf{w_h})\tau_1 \mathbf{r_m} + \int_{\Omega_e} \rho(\nabla \cdot \mathbf{w_h})\tau_2 r_c - \\ + \int_{\Omega_e} \rho \nabla q_h \tau_1 \mathbf{r_m} + \int_{\Omega_e} \rho \nabla q_h^* \tau_1 \mathbf{r_m} \quad (21)$$

This functional will be used for each element to obtain its sitfness matrix (lhs) and the loads vector (rhs). These terms will be later assembled to build the global system. The solution of the problem is such that the residuals of the momentum and mass equations are equal to zero. Adding the stabilization terms we can define the following residuals:

$$R_m^i = r_m^i(u_h, p_h^{tot}) + r_m^{i,stab}(u_h, p_h^{tot}) \quad (22)$$

$$R_c = r_c(u_h, p_h^{tot}) + r_c^{stab}(u_h, p_h^{tot}) \quad (23)$$

$$R_c^* = r_c(u_h, p_h^{tot}) + r_{c^*}^{stab}(u_h, p_h^{tot}) \quad (24)$$

These stabilization terms are:

$$r_m^{i,stab}(u_h, p_h^{tot}) = \partial_{w_i}(\rho(\mathbf{a} \cdot \nabla \mathbf{w}_h)\tau_1 \cdot \mathbf{r_m}(\mathbf{u_h}, p_h^{tot}) + \rho(\nabla \cdot \mathbf{w}_h)\tau_2 r_c(\mathbf{u_h})) \quad (25)$$

$$r_c^{stab}(u_h, p_h^{tot}) = \partial_q(\rho \nabla q_h \tau_1 \cdot \mathbf{r_m}(\mathbf{u_h}, p_h^{tot})) \quad (26)$$

$$r_{c^*}^{stab}(u_h, p_h^{tot}) = \partial_{q^*}(\rho \nabla q_h^* \tau_1 \cdot \mathbf{r_m}(\mathbf{u_h}, p_h^{tot})) \quad (27)$$

Take in acccount that as the terms that we added to the residuals that correspond to the stabilization depend on the residuals of the Navier Stokes equations, so if the residuals $\boldsymbol{r_m}$ and r_c are equal to zero, these new residuals R_m, R_c and R_{c*} will be equal to zero as well.

We want to impose that the residuals of the problem with the added stabilization terms are equal to zero. $R_m = 0$, $R_c = 0$ and $R_{c*} = 0$. We can see now that we can impose that from our functional by solving:

$$\frac{\partial f}{\partial v_i} = R_i = 0 \quad (28)$$

where v_i are the test functions of our problem: $\mathbf{w_h}, q_h$ and q_h^*. Using Newton-Raphson to solve equation 28 we get:

$$\frac{\partial R_i}{\partial u_j} \triangle u_j = R_i \quad (29)$$

or equivalently:

$$\frac{\partial^2 f}{\partial v_i \partial u_j} \triangle u_j = \frac{\partial f}{\partial v_i} \tag{30}$$

where u_j represents the unknowns of the problem: $\mathbf{u_h}, p_h$ and p_h^*. This allows us to identify the element contributions as:

$$lhs = \frac{\partial^2 f}{\partial v_i \partial u_j} = \frac{\partial R_i}{\partial u_j} \tag{31}$$

$$rhs = \frac{\partial f}{\partial v_i} = R_i \tag{32}$$

The system is therefore:

$$\begin{pmatrix} \frac{\partial R_m}{\partial u_h} & \frac{\partial R_m}{\partial p_h} & \frac{\partial R_m}{\partial p_h^*} \\ \frac{\partial R_c}{\partial u_h} & \frac{\partial R_c}{\partial p_h} & \frac{\partial R_c}{\partial p_h^*} \\ \frac{\partial R_{c^*}}{\partial u_h} & \frac{\partial R_{c^*}}{\partial p_h} & \frac{\partial R_{c^*}}{\partial p_h^*} \end{pmatrix} \begin{pmatrix} u_h \\ p_h \\ p_{h^*} \end{pmatrix} = \begin{pmatrix} R_m \\ R_c \\ R_{c^*} \end{pmatrix} \tag{33}$$

Written in a more compact way:

$$\begin{pmatrix} K & V \\ H & K_{ee} \end{pmatrix} \begin{pmatrix} \mathbf{u_h}, p_h \\ p_h^* \end{pmatrix} = \begin{pmatrix} b + f_v \\ f_e \end{pmatrix} \tag{34}$$

Where:

$$A_{11} = \begin{pmatrix} \frac{\partial R_m}{\partial u_h} & \frac{\partial R_m}{\partial p_h} \\ \frac{\partial R_c}{\partial u_h} & \frac{\partial R_c}{\partial p_h} \end{pmatrix} \qquad A_{12} = \begin{pmatrix} \frac{\partial R_m}{\partial p_h^*} \\ \frac{\partial R_c}{\partial p_h^*} \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} \frac{\partial R_{c^*}}{\partial u_h} & \frac{\partial R_{c^*}}{\partial p_h} \end{pmatrix} \qquad A_{22} = \begin{pmatrix} \frac{\partial R_{c^*}}{\partial p_h^*} \end{pmatrix}$$

$$f_1 = \begin{pmatrix} R_m \\ R_c \end{pmatrix}$$

$$f_2 = \begin{pmatrix} R_{c^*} \end{pmatrix}$$

Notice that the matrix A_{11} is the usual stiffness matrix that we would obtain if we did not use an enrichment for the pressure. We can condensate the degrees of freedom corresponding to $p_h{}^*$ obtaining.

$$(A_{11} - A_{12} A_{22}^{-1} A_{21})(\mathbf{u_h}, p_h) = f_1 - A_{12} A_{22}^{-1} f_2 \tag{35}$$

3 Discussion

The presented work shows the correct method of implementing an element of navier stokes within the scope of fixed mesh methods. The use of an enrichment in the pressure allows to overcome the problem of the discontinuity of the pressure gradients at the interface. The method set forth adds an additional advantage over conventional XFEM methods. The fact that the enrichment of the elements cut by the interface occurs only locally, causes that it is not necessary to add new degrees of freedom to the system. As our enriched shape functions functions vanish in the nodes, they do not contribute to the value of the pressure in those nodes.

References

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