Communication Skills 2

Assignment 2

Long Writing

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Integration of the hybridizable discontinuous Galerkin (HDG) method and NURBS-Enhanced Finite Element Method (NEFEM)

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Abstract

NEFEM is a technique that allows a seamless integration of the CAD boundary representation of the domain and the finite element method (FEM). The HDG is a new class of discontinuous Galerkin (DG) methods that shares favorable properties with classical mixed methods such as the well known Raviart-Thomas methods. In particular, HDG provides optimal convergence of both the primal and the dual variables of the mixed formulation. This property enables the construction of superconvergent solutions, contrary to other popular DG methods. NEFEM and HDG have been integrated to develop a high-order accurate space discretisation method for flow problems. The developed method was applied to the Poisson's problem, obtaining solutions that converged at optimal rates.

Keywords: finite elements, high-order, discontinuous Galerkin, HDG, NURBS, NEFEM

1 Introduction

The last few years have witnessed an increase in enthusiasm in the development of high-order methods within the finite element community. High-order approximations have proved to be of special importance while solving high Reynolds number and transient fluid problems [1]. The ability and efficiency of high-order approximations have been discussed and proven in recent papers [2]. Specifically, the interest in DG methods has increased over the past years. In particular, the HDG method, with its stability features, its reduced number of degrees of freedom, and its superconvergence properties has attracted attention among all DG methods for implicit schemes [3, 4, 5, 6, 7, 8].

However, in order to be competitive, these methods have to be designed in such a way that the increased associated computational complexity is more than balanced. Adaptive mesh refinement is a well known strategy for reducing the cost of a computational simulation while achieving a given level of accuracy [1, 9]. The superconvergence properties of the HDG method can come in handy while building in an automatic a posteriori error estimate based adaptive mesh refinement. However, this might result in higher geometric errors emerging from the change in the boundary representation due to a local p-adaptive mesh refinement. Hence in addition to efficiency and robustness, higher-order boundary representation also continues to be a major obstacle in the path of introducing high-order methods into industrial design processes [1].

For high-order methods to be viable in comparison with conventional FE techniques, it becomes quite essential to use considerably large elements while the variables of interest must have fewer degrees of freedom [10]. Also an interpolation polynomial of high degree must be adopted. However, the use of isoparametric elements will not be suitable in the adoption of the aforementioned parameters, as it will lead to unsustainable geometric errors that will do away with the benefits of using high-order methods [10]. NEFEM can easily be extended to high-order methods such as to the HDG method. The work presented in this report has been carried forward to establish a seamless integration between NEFEM and the HDG method. While HDG method comes with all the advantages of a high-order method, NEFEM helps tackle the issues of geometric error engendering out of the use of isoparametric elements. Combined, a method encompassing both NEFEM and the HDG method, seamlessly integrated with one another offers hope to overcome all of the previously mentioned obstacles lingering in the path of introducing high-order methods into industrial design processes.

2 HDG applied to Poisson's equation

Let $\Omega \in \mathbb{R}^{\texttt{n}_\texttt{sd}}$ be an open bounded domain with boundary $\partial\Omega$ and $\texttt{n}_\texttt{sd}$ the number of spatial dimensions. The strong from for Poisson's equation along with boundary conditions can be written as

$$\begin{cases} -\nabla \cdot \nabla u = f & \text{in } \Omega, \\ u = u_D & \text{on } \Gamma_D, \\ \boldsymbol{n} \cdot \nabla u = t & \text{on } \Gamma_N, \end{cases} \tag{1}$$

where $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N, \overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset, f \in \mathcal{L}_2(\Omega)$ is a source term and $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$. Here standard Dirichlet and Neumann boundary conditions are considered. Other mixed (i.e. Robin) boundary conditions can also be imposed [10]. The strong from is written in mixed form as a system of first order equations over the broken computational domain as

$$\begin{cases} \nabla \cdot \boldsymbol{q} = f & \text{in } \Omega_i \text{ and for } i = 1..., \texttt{n}_\texttt{el}, \\ \boldsymbol{q} + \nabla u = 0 & \text{in } \Omega_i \text{ and for } i = 1..., \texttt{n}_\texttt{el}, \\ u = u_D & \text{on } \Gamma_D, \\ \boldsymbol{n} \cdot \boldsymbol{q} = -t & \text{on } \Gamma_N, \\ [\![u\boldsymbol{n}]\!] = \boldsymbol{0} & \text{on } \Gamma, \\ [\![\boldsymbol{n} \cdot \nabla u]\!] = 0 & \text{on } \Gamma, \end{cases} \tag{2}$$

for $i = 1, ..., \texttt{n}_\texttt{el}$ and where $\boldsymbol{q} = -\nabla u$ is a new variable. The jump $[\![\cdot]\!]$ operator is defined such that, along each portion of the interface Γ it sums the values from the element on the left and right of say, Ω_i and Ω_j, namely

$$[\![\odot]\!] = \odot_i + \odot_j$$

The strong form is written in terms of the local problem and the global problem. First, the local, element-by-element, problem with Dirichlet boundary conditions is defined,

$$\begin{cases} \nabla \cdot \boldsymbol{q}_i = f & \text{in } \Omega_i, \\ \boldsymbol{q}_i + \nabla u_i = \boldsymbol{0} & \text{in } \Omega_i, \\ u_i = u_D & \text{on } \partial\Omega_i \cap \Gamma_D, \\ u_i = \hat{u} & \text{on } \partial\Omega_i \backslash \Gamma_D, \end{cases} \tag{3}$$

for $i = 1, ..., \texttt{n}_{\texttt{el}}$. It is considered that $\hat{u} \in \mathcal{L}_2(\Gamma \cup \Gamma_N)$ is given. This problem produces an element-by-element solution in each element Ω_i for q_i and u_i as a function of the unknown $\hat{u}$. As a result of this, these problems can be solved independently element by element.

Second, a global problem is defined to determine $\hat{u}$ which corresponds to the imposition of the Neumann boundary condition.

$$\begin{cases} [\![u\boldsymbol{n}]\!] = \boldsymbol{0} & \text{on } \Gamma, \\ [\![\boldsymbol{n} \cdot \boldsymbol{q}]\!] = 0 & \text{on } \Gamma, \\ \boldsymbol{n} \cdot \boldsymbol{q} = -t & \text{on } \Gamma_N, \end{cases} \tag{4}$$

The numerical traces of the fluxes are defined element-by-element (i.e. for $i = 1, ..., \texttt{n}_{\texttt{el}}$) for the sake of stability as

$$\boldsymbol{n}_i \cdot \widehat{\boldsymbol{q}}_i := \begin{cases} \boldsymbol{n}_i \cdot \boldsymbol{q}_i + \tau_i(u_i - u_D) & \text{on } \partial\Omega_i \cap \Gamma_D, \\ \boldsymbol{n}_i \cdot \boldsymbol{q}_i + \tau_i(u_i - \hat{u}) & \text{elsewhere}, \end{cases} \tag{5}$$

with τ_i being a stabilization parameter defined element-by-element, whose selection has an important effect on the stability, accuracy and convergence properties of the resulting HDG method.

Necessary discrete spaces required to prescribe the discrete weak forms for the local and global problems are chosen. With the definition of the numerical fluxes given by Equation (5), the discretized weak form of the local problems are: for $i = 1, ..., \texttt{n}_{\texttt{el}}$, find $(\boldsymbol{q}_i^h, u_i^h) \in \boldsymbol{\mathcal{W}}^h \times \mathcal{V}^h$ for all $(\boldsymbol{w}, v) \in \boldsymbol{\mathcal{W}}(\Omega_i) \times \mathcal{V}(\Omega_i)$ such that

$$\overbrace{< v, \tau_i u_i^h >_{\partial\Omega_i}}^{\boldsymbol{A}_{uu}\mathbf{u}_i} \overbrace{-(v, \nabla \cdot \boldsymbol{q}_i^h)_{\Omega_i}}^{\boldsymbol{A}_{uq}\mathbf{q}_i} = \overbrace{(v, f)_{\Omega_i} + < v, \tau_i u_D >_{\partial\Omega_i \cap \Gamma_D}}^{\mathbf{f}_u} + \overbrace{< v, \tau_i \hat{u}^h >_{\partial\Omega_i \setminus \Gamma_D}}^{\boldsymbol{A}_{u\hat{u}}\hat{\mathbf{u}}_i}, \tag{6a}$$

$$\overbrace{(\nabla \cdot \boldsymbol{w}, u_i^h)_{\Omega_i}}^{\boldsymbol{A}_{uq}^T\mathbf{u}_i} \overbrace{-(\boldsymbol{w}, \boldsymbol{q}_i^h)_{\Omega_i}}^{\boldsymbol{A}_{qq}\mathbf{q}_i} = \overbrace{< \boldsymbol{n}_i \cdot \boldsymbol{w}, u_D >_{\partial\Omega_i \cap \Gamma_D}}^{\mathbf{f}_q} + \overbrace{< \boldsymbol{n}_i \cdot \boldsymbol{w}, \hat{u}^h >_{\partial\Omega_i \setminus \Gamma_D}}^{\boldsymbol{A}_{q\hat{u}}\hat{\mathbf{u}}_i}, \tag{6b}$$

whereas the global problem is: find $\hat{u}^h \in \mathcal{M}^h(\Gamma \cup \Gamma_N)$ for all $\mu \in \mathcal{M}^h(\Gamma \cup \Gamma_N)$ such that

$$\sum_{i=1}^{\texttt{n}_{\texttt{el}}} \{ \overbrace{< \mu, \tau_i u_i^h >_{\partial\Omega_i \setminus \Gamma_D}}^{\boldsymbol{A}_{u\hat{u}}^T\mathbf{u}_i} + \overbrace{< \mu, \boldsymbol{n}_i \cdot \boldsymbol{q}_i^h >_{\partial\Omega_i \setminus \Gamma_D}}^{\boldsymbol{A}_{q\hat{u}}^T\mathbf{q}_i} + \overbrace{< \mu, \tau_i \hat{u}^h >_{\partial\Omega_i \setminus \Gamma_D}}^{\boldsymbol{A}_{\hat{u}\hat{u}}\hat{\mathbf{u}}_i} \} = \sum_{i=1}^{\texttt{n}_{\texttt{el}}} \overbrace{- < \mu, t >_{\partial\Omega_i \cap \Gamma_N}}^{\mathbf{f}_{\hat{u}}} \tag{7}$$

With the chosen interpolation, equations (6) result in a system of equations for each element Ω_i while the same interpolation applied to Equations (7) produce a system of

(a) (b)

Figure 1: Coarse mesh of the domain (Ω) with nodes for $k = 2$ (a) and model problem solution for the same value of k for the mesh shown in (a).

global equations. See [11] for the detailed derivation of HDG for Poisson's system. The concept of NEFEM was introduced from the classical FEM point of view. Its extension to the HDG method follows the same path. Only those elements in the mesh that are affected by curved boundaries will be subjected to this integration, while the other elements with planar boundaries will go through the usual HDG routines for the computation of elemental matrices and vectors. See [10] for details of NEFEM.

3 Results

In order to explain the results of integration of NEFEM and HDG for Poisson's system, the problem is is solved in a domain bound by the lines $x = 0$, $y = 0$ and $y = 1$ on the left, bottom and top, respectively, and by the NURBS curve $(x-1)^2+(y-0.5)^2 = 0.5^2$ (this is the implicit, and not the parametrized equation of the NURBS curve) on the right, with $\Gamma_N = \{(x, y) \in \partial\Omega \mid (x-1)^2+(y-0.5)^2 = 0.5^2 \ , \ y \leq 0.5\}$ and $\Gamma_D = \partial\Omega\backslash\Gamma_N$. The source and the boundary conditions are taken such that the analytical solution is given by

$$u(x, y) = x\cos(y) + y\sin(x)$$

where, u is the velocity of the fluid particle at a point whose coordinates are (x, y).

The plot shown in Figure 1a shows the mesh that was considered for the test, a coarse mesh with only 12 elements. The approximation order considered for NEFEM integrated to HDG computation is $k = 2$, and the black dots on the triangles denote the nodes used to build the polynomial approximation of the primal and the dual variables, u^h and $\boldsymbol{q}^h$ respectively. The numerical solution computed with a polynomial approximation of degree $k = 2$ is depicted in Figure 1b. From the plot, the discontinuity of the numerical solution, u^h, which an inherent property of the DG methods, is not very obvious. This plot is not able to capture the discontinuity because, either $k = 2$ is good enough to capture the solution for the given domain, or 2D plots are not good enough to capture the discontinuities.

Finally, an h-convergence study is performed in order to check the optimal approximation properties of the implemented integration of NEFEM with HDG. Figure 2 shows the evolution of the error of u^h and $u^{\star h}$ in the $\mathcal{L}_2(\Omega)$ norm as a function of the characteristic element size h for a degree of approximation, k, ranging from 1 to 3. For all the degrees of approximation considered, the optimal rate of convergence (i.e., $k+1$ for the solution and $k+2$ for the postprocessed solution) is obtained.

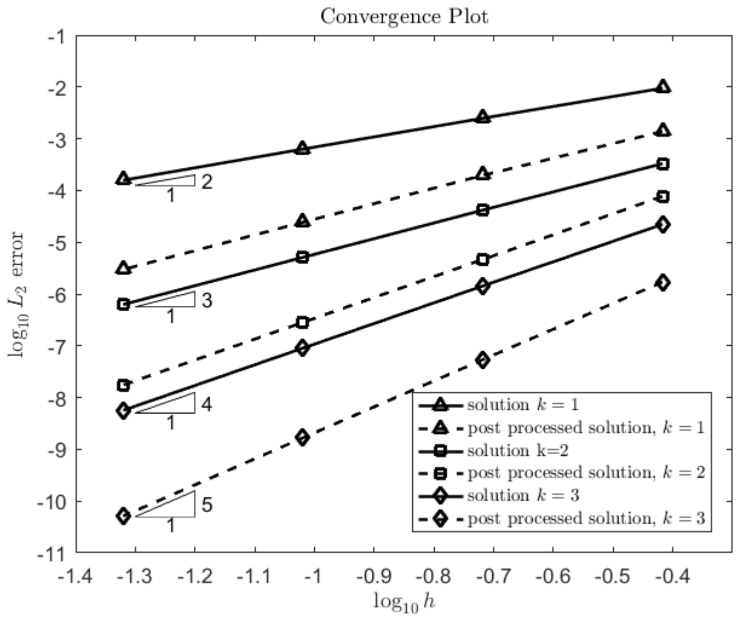

Figure 2: Convergence plots for $k = 1, 2, 3$.

4 Concluding remarks

The main purpose of this work was to gain a working understanding of the HDG method and NEFEM, two relatively new approaches in the area of computational methods based on finite elements, in order to integrate these two cutting-edge techniques seamlessly into one working code for the analysis of problems of interest to the aerodynamic aircraft design industry. This integrated method was implemented to the 2-D Poisson's equation. The obtained results are in agreement with the expected solution, with errors converging at optimal rates. In the process of doing this a number of interesting observations were made that have a opened a number of research lines.

NEFEM integrated to HDG presents a case of a high-order method that promises to be devoid of the issues that are preventing the industry from embracing high-order methods. Further research needs to be done in order to finally extend this method to the case of incompressible and compressible Navier-Stokes' equations, in order to prove the competency of the method. The extension of the carried work to the 3-D case and implementation of adaptive mesh refinement are some of the work that need immediate attention and will be the areas of focus in the work that will follow.

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